

Model Ambiguity in Risk Sharing with Monotone Mean-Variance

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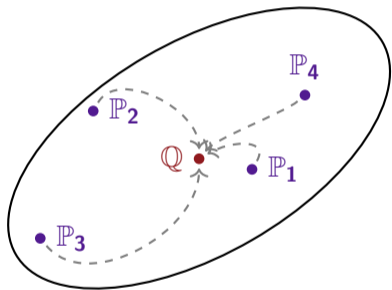
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Introduction



An agent has multiple models/probability measures P_1, P_2, P_3, P_4 .

The agent has to make a decision optimally accounting for ambiguity about these models.

Agent must choose a model Q to optimize under.

In our setting:

- ▶ agent = **insurer**
- ▶ decision = **risk sharing**
- ▶ penalization = **chi-squared divergence**
(monotone mean variance)

Model combination and model ambiguity

Let $D(\cdot, \cdot)$ be a divergence measure, ex., Kullback-Leibler divergence.

Let \mathbb{P} be a probability measure.

Model ambiguity

Model combination (barycenter)

$$\inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}[U(X_T) + \frac{1}{\varepsilon} D(\mathbb{Q}, \mathbb{P})]$$

where $\varepsilon > 0$.

e.g., [Hansen and Sargent, 2008] [Maccheroni et al., 2006] [Maccheroni et al., 2009]

The monotone mean-variance criterion [Maccheroni et al., 2009]

$$J_{\theta}^{MV}[X] = \mathbb{E}^{\mathbb{P}}[X] - \frac{\theta}{2} \text{Var}^{\mathbb{P}}(X)$$

$$J_{\theta}^{MMV}[X] := \min_{\mathbb{Q} \in \Delta^2(\mathbb{P})} \left(\mathbb{E}^{\mathbb{Q}}[X] + \frac{1}{2\theta} \mathbb{E}^{\mathbb{P}} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)^2 - 1 \right] \right)$$

where $\Delta^2(\mathbb{P}) = \{ \mathbb{Q} \ll \mathbb{P} : \mathbb{E}^{\mathbb{P}} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)^2 \right] < \infty \}$.

Properties of MMV [Maccheroni et al., 2009]

- ▶ Agrees with MV criterion where it is monotone
- ▶ Best possible monotone approximation of the MV criterion outside of where it is monotone
- ▶ Unlike MV, MMV preserves second-order stochastic dominance

Model combination and model ambiguity

Let $D(\cdot, \cdot)$ be a divergence measure, ex., Kullback-Leibler divergence.

Let $\mathbb{P}, \mathbb{P}_k, k = 1, \dots, n$, be probability measures.

Model ambiguity	Model combination (barycenter)
$\inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}[U(X_T) + \frac{1}{\varepsilon} D(\mathbb{Q}, \mathbb{P})]$	$\inf_{\mathbb{Q}} \sum_{k=1}^n \pi_k D(\mathbb{Q}, \mathbb{P}_k)$
where $\varepsilon > 0$.	where $\pi_k \geq 0, \sum_{k=1}^n \pi_k = 1$.
e.g., [Hansen and Sargent, 2008] [Maccheroni et al., 2006] [Maccheroni et al., 2009]	[Liu et al., 2025] [Acciaio et al., 2025] [Jaimungal and Pesenti, 2025] [Kroell et al., 2025]

Our problem:

$$\inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[U(X_T) + \frac{1}{\varepsilon} \sum_{k=1}^n \pi_k D(\mathbb{Q}, \mathbb{P}_k) \right]$$

Setting

- ▶ **Insurer** in a non-life insurance market faces insurance losses over a finite horizon $[0, T]$.
- ▶ **Insurer** can share their risk with another agent, the **counterparty**, by ceding them a portion of their loss in return for a premium payment.
- ▶ Insurer has multiple models for the loss distribution: $\mathbb{P}_1, \dots, \mathbb{P}_n, \mathbb{P}_C$ and chooses a model \mathbb{Q} to optimize the risk sharing under; counterparty sets premium under their model, \mathbb{P}_C .

Probabilistic set-up

- ▶ Assume a complete, filtered measurable space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]})$ and $n + 1$ equivalent probability measures $\mathbb{P}_1, \dots, \mathbb{P}_n, \mathbb{P}_C$.
- ▶ $N(d\xi, dt)$ is a Poisson random measure driving the insurance losses in the market.
- ▶ Under a measure \mathbb{P}_k for $k \in \mathcal{I}$, $\mathcal{I} := \{1, \dots, n, C\}$, N has \mathbb{P}_k -compensator $\nu_k(d\xi, dt) = \nu_k(d\xi)dt$.
- ▶ Define the \mathbb{P}_k -compensated PRM by

$$\tilde{N}^{\mathbb{P}_k}(d\xi, dt) = N(d\xi, dt) - \nu_k(d\xi)dt.$$

- ▶ Each compensator admits a density $\nu_k(\xi)$, i.e., $\nu_k(d\xi) = \nu_k(\xi)d\xi$ for $k \in \mathcal{I}$.

Assumption

$$\int_{\mathbb{R}_+} \frac{\nu_C^2(\xi)}{\nu_k(\xi)} d\xi < \infty \text{ for } k \in \mathcal{I}, \quad \int_{\mathbb{R}_+} \frac{\nu_C^3(\xi)}{\nu_j(\xi)\nu_k(\xi)} d\xi < \infty \text{ for } j, k \in \mathcal{I}.$$

Insurer's surplus

- ▶ The insurer's wealth process follows a Cramér-Lundberg model with constant premium rate $c > 0$ and initial wealth $x_0 > 0$:

$$X_t^{CL} = x_0 + ct - \int_0^t \int_{\mathbb{R}_+} \xi N(d\xi, ds).$$

- ▶ Insurer cedes a portion $\alpha_t(\xi)$ of the loss $\xi \in \mathbb{R}_+$ to the counterparty.

Definition (admissible risk sharing strategies)

We define the set of admissible risk sharing strategies, \mathcal{A} , as those strategies α_t that are \mathbb{F} -predictable random fields satisfying for $t \in [0, T]$,

$$\mathbb{E}^{\mathbb{P}^c} \left[\int_0^t \int_{\mathbb{R}_+} |\alpha_s(\xi)|^2 \nu_C(d\xi) ds \right] < \infty \quad \text{and}$$

$$\mathbb{E}^{\mathbb{P}^c} \left[\int_0^t \int_{\mathbb{R}_+} [\xi - \alpha_s(\xi)]^2 \nu_C(d\xi) ds \right] < \infty.$$

Insurer's surplus

- ▶ The counterparty charges the expected value premium principle with safety loading $\eta > 0$: $(1 + \eta) \int_{\mathbb{R}_+} \alpha_t(\xi) \nu_C(d\xi)$.
- ▶ Assume that the risk sharing premium is such that $c < (1 + \eta) \int_{\mathbb{R}_+} \xi \nu_C(d\xi)$.
- ▶ The insurer's wealth process $X^\alpha := (X_t^\alpha)_{t \in [0, T]}$ is

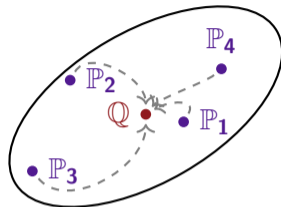
$$X_t^\alpha = x_0 + \int_0^t \left[\underbrace{c}_{\text{insurer's premium}} - \underbrace{(1 + \eta) \int_{\mathbb{R}_+} \alpha_u(\xi) \nu_C(d\xi)}_{\text{counterparty's premium}} \right] du - \underbrace{\int_0^t \int_{\mathbb{R}_+} [\xi - \alpha_u(\xi)] N(d\xi, du)}_{\text{losses retained by insurer}} .$$

Insurer's criterion

Recall: the insurer has $n + 1$ models/probability measures $\mathbb{P}_1, \dots, \mathbb{P}_n, \mathbb{P}_C$.

Insurer penalizes model ambiguity using the χ^2 -divergence:

$$\chi^2(\mathbb{Q} \parallel \mathbb{P}) := \mathbb{E}^{\mathbb{P}} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)^2 - 1 \right].$$



Optimization Problem

The insurer seeks the solution to the following problem:

$$\sup_{\alpha \in \mathcal{A}} \inf_{\mathbb{Q} \in \Delta^2} \left(\mathbb{E}^{\mathbb{Q}}[X_T^\alpha] + \frac{1}{2\theta} \sum_{k \in \mathcal{I}} \pi_k \mathbb{E}^{\mathbb{P}_k} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}_k} \right)^2 - 1 \right] \right),$$

where $\theta > 0$, $\pi_k \geq 0$, $k \in \mathcal{I} := \{1, \dots, n, C\}$, are given weights such that

$\sum_{k \in \mathcal{I}} \pi_k = 1$, and $\Delta^2 := \{ \mathbb{Q} : \mathbb{Q} \ll \mathbb{P}_k \text{ and } \mathbb{E}^{\mathbb{P}_k} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}_k} \right)^2 \right] < \infty \text{ for all } k \in \mathcal{I} \}$.

Auxiliary processes

Definition (auxiliary processes)

For an \mathbb{F} -predictable random field $\beta = (\beta_t)_{t \in [0, T]}$, $\beta_t : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, define the stochastic processes $\{Z_{k,t}^\beta\}_{t \in [0, T], k \in \mathcal{I}}$, $k \in \mathcal{I}$, which solve the SDEs:

$$dZ_{k,t}^\beta = Z_{k,t^-}^\beta \int_{\mathbb{R}_+} [v_k(\xi) - \beta_t(\xi)] d\xi dt - Z_{k,t^-}^\beta \int_{\mathbb{R}_+} \left[1 - \frac{\beta_t(\xi)}{v_k(\xi)} \right] N(d\xi, dt),$$
$$Z_{k,0}^\beta = 1.$$

Define \mathbb{Q}_β such that $d\mathbb{Q}_\beta(\omega) = Z_{k,T}^\beta d\mathbb{P}_k(\omega)$ for all $\omega \in \mathcal{F}_T$, $k \in \mathcal{I}$.

Then by Girsanov's theorem:

$$\tilde{N}^{\mathbb{Q}_\beta}(d\xi, dt) = \tilde{N}^{\mathbb{P}_k}(d\xi, dt) + \left[1 - \frac{\beta_t(\xi)}{v_k(\xi)} \right] v_k(d\xi) dt = N(d\xi, dt) - \beta_t(\xi) d\xi dt,$$

i.e. N has \mathbb{Q}_β -compensator $\beta_t(\xi) d\xi dt$.

Re-writing the criterion

Definition (admissible compensators)

Let \mathfrak{B} denote the \mathbb{F} -predictable random fields β such that, for all $k \in \mathcal{I}$,

$$\mathbb{E}^{\mathbb{P}_k} \left[\left(Z_{k,T}^\beta \right)^2 \right] < \infty.$$

Lemma

The processes $\beta \in \mathfrak{B}$ induce the same probability measures as

$$\Delta^2 = \left\{ \mathbb{Q} : \mathbb{Q} \ll \mathbb{P}_k \text{ and } \mathbb{E}^{\mathbb{P}_k} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}_k} \right)^2 \right] < \infty \text{ for all } k \in \mathcal{I} \right\}, \text{ i.e.,}$$

$$\Delta^2 = \{ \mathbb{Q}_\beta : \beta \in \mathfrak{B} \}.$$

Insurer's optimization problem

Optimization Problem

The insurer seeks the solution to the following problem:

$$\sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \mathbb{E}^{\mathbb{Q}_\beta} \left[X_T^\alpha + \frac{1}{2\theta} \sum_{k \in \mathcal{I}} \pi_k \left(Z_{k,T}^\beta - 1 \right) \right],$$

where $\theta > 0$ and $\pi_k \geq 0$, $k \in \mathcal{I} := \{1, \dots, n, C\}$, are given weights such that $\sum_{k \in \mathcal{I}} \pi_k = 1$.

Theorem (Optimal Controls)

The optimal controls in feedback form are

$$\alpha^*(t, \xi, \mathbf{z}) = \xi - \frac{1}{\theta} \sum_{k \in \mathcal{I}} \pi_k z_k \ell_k(T - t) \left[(1 + \eta) \frac{v_C(\xi)}{v_k(\xi)} - 1 \right],$$
$$\beta^*(\xi) = (1 + \eta) v_C(\xi),$$

where

$$\ell_k(t) = \exp \left(t \int_{\mathbb{R}_+} \left[1 - (1 + \eta) \frac{v_C(\xi)}{v_k(\xi)} \right]^2 \nu_k(d\xi) \right),$$

and the insurer's value function is

$$J(t, x, \mathbf{z}) = x + \sum_{k \in \mathcal{I}} \frac{\pi_k}{2\theta} z_k \ell_k(T - t) - \frac{1}{2\theta} - \left[(1 + \eta) \int_{\mathbb{R}_+} \xi \nu_C(d\xi) - c \right] (T - t).$$

Processes under optimal controls

Proposition

For $t \in [0, T]$,

$$X_t^* = x_0 + \left[c - (1 + \eta) \int_{\mathbb{R}_+} \xi \nu_C(d\xi) \right] t + \frac{1}{\theta} \sum_{k \in \mathcal{I}} \pi_k \ell_k(T) [1 - \ell_k(-t) Z_{k,t}^*] .$$

where

$$Z_{k,t}^* = \exp \left(t \int_{\mathbb{R}_+} [v_k(\xi) - (1 + \eta)v_C(\xi)] d\xi + \int_0^t \int_{\mathbb{R}_+} \ln \left((1 + \eta) \frac{v_C(\xi)}{v_k(\xi)} \right) N(d\xi, ds) \right), k \in \mathcal{I} .$$

Sketch of proof: optimal controls

α^* , β^* , J derived using the Hamilton-Jacobi-Bellman-Isaacs equation.

Are α^* and β^* admissible?

Lemma

For $k \in \mathcal{I}$: $\mathbb{E}^{\mathbb{P}^k}[Z_{k,T}^*] = 1$, $\mathbb{E}^{\mathbb{P}^k}[(Z_{k,T}^*)^2] = \ell_k(T) < \infty$, $\mathbb{E}^{\mathbb{P}^c}[(Z_{k,T}^*)^2] < \infty$.

Then $\beta^* \in \mathfrak{B}$ and we can show that:

$$\left. \begin{aligned} \mathbb{E}^{\mathbb{P}^c} \left[\int_0^t \int_{\mathbb{R}_+} |\alpha^*(s, \xi, \mathbf{z}_s^*)|^2 \nu_C(d\xi) ds \right] < \infty \\ \mathbb{E}^{\mathbb{P}^c} \left[\int_0^t \int_{\mathbb{R}_+} [\xi - \alpha^*(s, \xi, \mathbf{z}_s^*)]^2 \nu_C(d\xi) ds \right] < \infty \end{aligned} \right\} \alpha^* \in \mathcal{A}$$

Mean and variance of insurer's wealth

Proposition

For $t \in [0, T]$:

$$\mathbb{E}^{\mathbb{Q}^*} [X_t^*] = x_0 + \left[c - (1 + \eta) \int_{\mathbb{R}_+} \xi \nu_C(d\xi) \right] t = \mathbb{E}^{\mathbb{Q}^*} [X_t^{CL}],$$

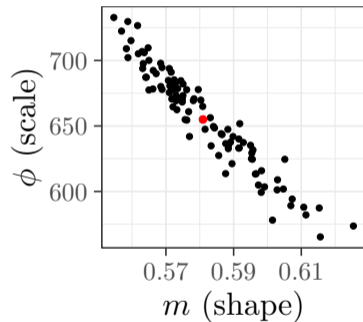
$$\text{Var}^{\mathbb{Q}^*} (X_t^*) = \frac{1}{\theta^2} \mathbf{p}_t^\top \Sigma_{\mathbf{Z}^*}^{\mathbb{Q}^*} \mathbf{p}_t.$$

where $\Sigma_{\mathbf{Z}^*}^{\mathbb{Q}^*}$ denotes the covariance matrix of \mathbf{Z}^* , i.e., $(\Sigma_{\mathbf{Z}^*}^{\mathbb{Q}^*})_{jk} = \text{Cov}^{\mathbb{Q}^*} (Z_{j,t}^*, Z_{k,t}^*)$ and $\mathbf{p}_t := (\pi_1 \ell_1(T-t), \dots, \pi_n \ell_n(T-t), \pi_C \ell_C(T-t))$.

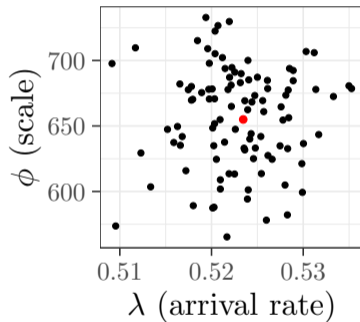
Motivating data example

- ▶ Recent open-access insurance data set [Segura-Gisbert et al., 2024], 105,555 observations, giving policy-level data on annual **motor insurance policies** of a Spanish non-life insurer for policies started in the years 2015–2018
- ▶ Using cross-validation, estimate 100 models \mathbb{P}_k , $k = 1, \dots, 100$ from the data set. For each estimate, we sample 50% of the data and then estimate the parameters.
- ▶ Estimate arrival rate and severity distribution by maximum likelihood.
- ▶ Assume that under all models $k \in \mathcal{I}$:
 - ▶ the claim arrival rate is Poisson distributed with rate $\lambda_k > 0$,
 - ▶ the severity distribution is Gamma distributed with shape parameter $m_k > 0$ and scale parameter $\phi_k > 0$.
- ▶ Estimate the counterparty's model, \mathbb{P}_C , using the full dataset.

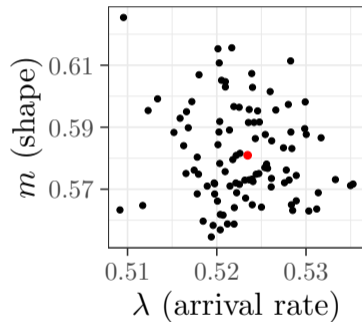
Estimated parameters



(a) Shape versus scale

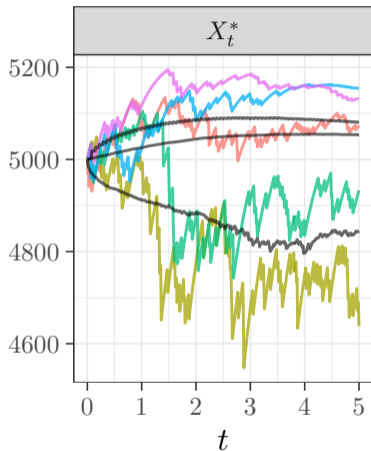


(b) Arrival rate versus scale

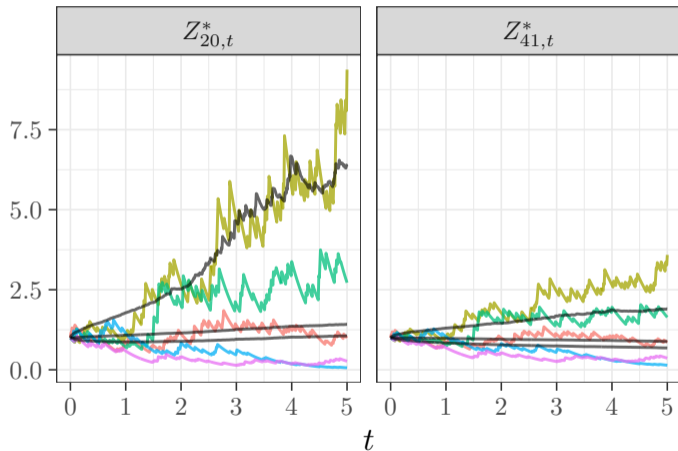


(c) Arrival rate versus shape

Paths of X^* and selected Z^* under \mathbb{P}_C

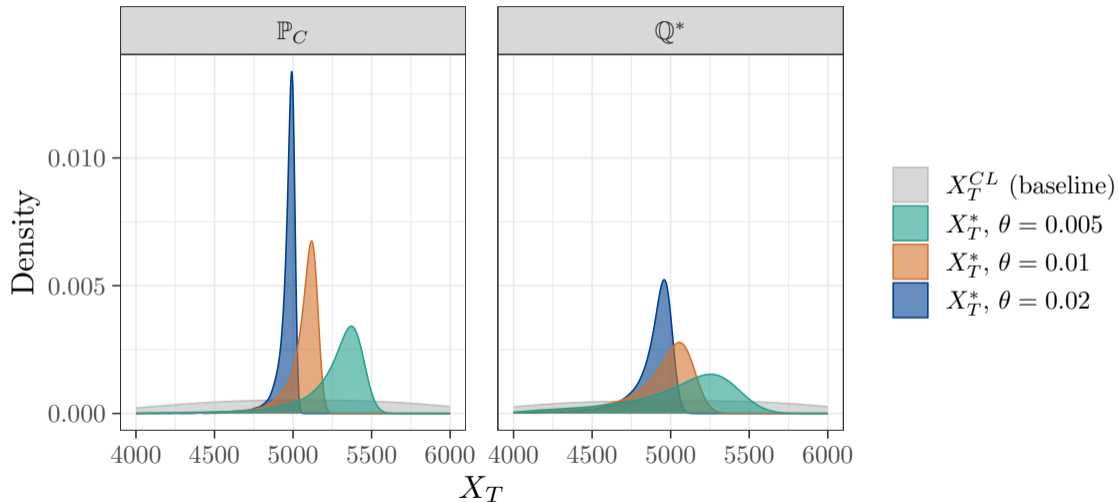


(a) Paths of X_t^* under \mathbb{P}_C



(b) Paths of $Z_{20,t}^*$, $Z_{41,t}^*$ under \mathbb{P}_C

KDE of X_T under different scenarios



Comparison with a risk-sharing mean-variance strategy

- ▶ If there is only one model, \mathbb{P} , then the MMV criterion with model ambiguity reduces to the original MMV criterion [Maccheroni et al., 2009]
- ▶ In continuous-time, the optimal strategies for MMV and MV generally coincide [Du and Strub, 2024; Y. Li et al., 2025; Trybuła and Zawisza, 2019]
- ▶ How does the optimal strategy for the risk sharing problem with model combination relate to the mean-variance strategy?

Criterion with one reference measure \mathbb{P} :

$$\sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathfrak{B}} \mathbb{E}^{\mathbb{Q}^\beta} \left[X_T^\alpha + \frac{1}{2\theta} (Z_T^\beta - 1) \right]$$

Monotone mean-variance problem

Proposition

The optimal controls for the problem with a single reference measure \mathbb{P} are

$$\begin{aligned}\alpha^*(t, \xi, Z_t) &= \xi - \frac{\eta}{\theta} e^{\lambda\eta^2(T-t)} Z_t, \\ \beta^*(\xi) &= (1 + \eta) v(\xi).\end{aligned}$$

Furthermore, the insurer's wealth under the optimal strategy is for $t \in [0, T]$,

$$X_t^* = x_0 - [(1 + \eta)\lambda\mu - c] t + \frac{1}{\theta} e^{\lambda\eta^2 T} - \frac{1}{\theta} e^{\lambda\eta^2(T-t)} Z_t^*.$$

where $Z_t^* = (1 + \eta)^{M_t} e^{-\eta\lambda t}$ and $M_t = \int_0^t \int_{\mathbb{R}_+} N(d\xi, dt)$.

Mean-variance problem

Optimization Problem 2 (Mean-Variance)

The insurer seeks the solution to the following problem for $\theta > 0$:

$$\sup_{\alpha \in \mathcal{A}} \left\{ \mathbb{E}^{\mathbb{P}} [X_T^\alpha] - \frac{\theta}{2} \text{Var}^{\mathbb{P}}(X_T^\alpha) \right\},$$

where

$$dX_t^\alpha = \left(c - \int_{\mathbb{R}_+} [\xi + \eta \alpha_t(\xi)] \nu(d\xi) \right) dt - \int_{\mathbb{R}_+} [\xi - \alpha_t(\xi)] \tilde{M}^{\mathbb{P}}(d\xi, dt), \quad X_0^\alpha = x_0.$$

Proposition

The optimal risk sharing strategy $\hat{\alpha}^*$ for Optimization Problem 2 is

$$\hat{\alpha}^*(t, \xi, \hat{X}_t^*) = \xi + \eta \left(\hat{X}_t^* - x_0 + t [(1 + \eta)\lambda\mu - c] - \frac{1}{\theta} e^{\lambda\eta^2 T} \right),$$

where \hat{X}_t^* is the insurer's wealth under the strategy $\hat{\alpha}^*$ with $\hat{X}_0^* = x_0$.

Proof follows by the pre-commitment approach of [Zhou and D. Li, 2000].

Proposition

The optimal risk sharing strategy for Optimization Problem 2, $\hat{\alpha}^*$, coincides with the optimal risk sharing strategy for Optimization Problem 1 with one reference measure \mathbb{P} , α^* .

Contributions

- ▶ We introduce a **novel model combination criterion** that generalizes the monotone mean-variance preferences to multiple reference models
- ▶ We derive **explicit solutions** for the insurer's **optimal risk-sharing strategy**, **optimal decision measure**, and their **wealth process**
- ▶ We prove that the strategy we obtain is admissible and that the value function satisfies the appropriate verification conditions
- ▶ We show how the optimal strategy relates to the pre-commitment mean variance strategy
- ▶ We determine the **mean** and **variance** of the insurer's wealth process X , and show that the model penalization parameter θ penalizes the variance of X
- ▶ We illustrate the strategy with recent open-access non-life insurance data

Thank you for your attention!

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