Optimal robust reinsurance with multiple insurers

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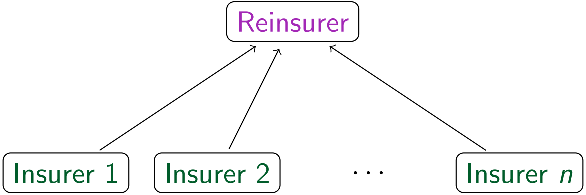
- Stackelberg game: reinsurer is the leader and the insurers are the followers



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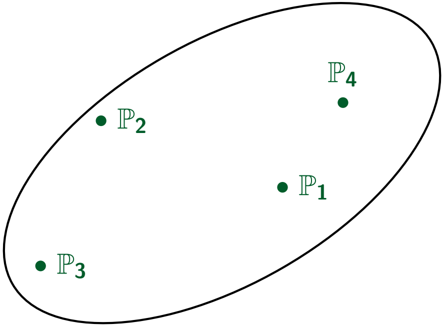

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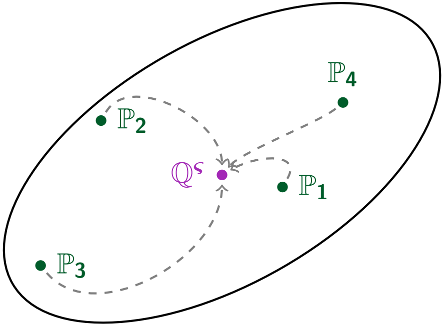
- **Reinsurer** maximizes their expected wealth under a **probability measure** $\mathbb{Q}^\varsigma$, which accounts for the different insurers' models as well as uncertainty about their accuracy



Background

- Stackelberg games in reinsurance setting introduced by [CS18], [CS19]
- Adding ambiguity aversion as a scaled KL-penalty [HCW18a], [HCW18b]
- Reinvestment of profits by the reinsurer: [GVS20], [GLS23]
- Ambiguity where insurer and reinsurer maximize their expected wealth [Cao+22a], [Cao+22b]
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Our setting:

- ***n* insurers** who maximize expected utility and are **ambiguity neutral**
- A single **reinsurer** who maximizes expected wealth and is **ambiguity averse**

Reinsurance contracts

Definition (Reinsurance Contract)

A reinsurance contract is characterised by a **retention function** $r\colon \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}_+$, which is:

- non-decreasing in the first argument,
- satisfies $0 < r(z, \boldsymbol{a}) \leq z$, for all $z \in \mathbb{R}_+$, $\boldsymbol{a} \in \mathcal{A}$,

and a corresponding **reinsurance premium** $p^R\colon \mathcal{A} \times \mathbb{R}_+ \to \mathbb{R}_+$.

For a tuple $(\boldsymbol{a}, \boldsymbol{c}) \in \mathcal{A} \times \mathbb{R}_+$, the reinsurer agrees to cover $z - r(z, \boldsymbol{a})$ for a premium $p^R(\boldsymbol{a}, \boldsymbol{c})$, where $\boldsymbol{c}$ is the reinsurer's safety loading.

Assumption

We consider retention functions that are continuous in the loss z and increasing and almost everywhere differentiable in the control $\boldsymbol{a}$.

Reinsurance contracts

Example (Proportional reinsurance)

The insurer chooses the proportion $\boldsymbol{a}$ of the loss to retain:

$$r(z, \boldsymbol{a}) = \boldsymbol{a} z, \quad \boldsymbol{a} \in (0, 1] \, .$$

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Example (Excess-of-loss insurance)

The insurer chooses the retention limit $\boldsymbol{a}$, beyond which the reinsurer covers any excess losses:

$$r(z, \boldsymbol{a}) = \min\{\boldsymbol{a}, z\}, \quad \boldsymbol{a} \in \mathbb{R}_+ \, .$$

Insurers

- Each insurer's loss process follows a Cramér-Lundberg model
  - Claims of insurer-k arrive according to a Poisson process with intensity $\lambda_k \in \mathbb{R}_+$
  - Claim severity $\sim F_k(\cdot)$ non-negative
- Insurer's premium rate is given by the expected value principle with safety loading $\theta_k > 0$
- The insurer's control of the retention function $\boldsymbol{\alpha_k} := (\boldsymbol{\alpha_{t,k}})_{t\in[0,T]}$ varies in time
- The k-th insurers' wealth process $X_k := (X_{t,k})_{t\in[0,T]}$ is

$$X_{t,k} = X_{0,k} + \int_0^t \left[p_k^I - p_k^R(\boldsymbol{\alpha_{u,k}}, \boldsymbol{c_k}) \right] du - \int_0^t \int_0^\infty r(z, \boldsymbol{\alpha_{u,k}}) N(dz, du) \, .$$

Reinsurer

- Reinsurer sets the reinsurance premium rate for insurer-k using the expected value principle with deterministic safety loading $\boldsymbol{\eta_k} := (\boldsymbol{\eta_{t,k}})_{t\in[0,T]}$:

$$p_k^R(\boldsymbol{\alpha_k}, \boldsymbol{\eta_{t,k}}) := (1 + \boldsymbol{\eta_{t,k}})\,\lambda_k \int_0^\infty \big[z - r(z, \boldsymbol{\alpha_{t,k}})\big]\, F_k(dz)\,.$$

- The reinsurer's wealth process $Y := (Y_t)_{t\in[0,T]}$ is

$$Y_t = Y_0 + \underbrace{\sum_{k\in\mathcal{N}} \int_0^t p_k^R(\boldsymbol{\alpha_{u,k}}, \boldsymbol{\eta_{u,k}})\, du}_{\text{aggregate premia}} - \underbrace{\int_0^t \int_0^\infty \sum_{k\in\mathcal{N}} \big[z - r(z, \boldsymbol{\alpha_{u,k}})\big]\, N(dz, du)}_{\text{aggregate losses}}\,.$$

Insurer's Problem

Insurer-k's Optimization Problem

Insurer-k seeks the contract parameters that attain the supremum

$$\sup_{\boldsymbol{\alpha_k} \in \mathfrak{A}} \mathbb{E}^{\mathbb{P}_k}\left[-\frac{1}{\gamma_k} e^{-\gamma_k X_{T,k}}\right].$$

Proposition

For $k \in \mathcal{N}$ and $\boldsymbol{c} \in \mathbb{R}_+$ consider the following non-linear equation for $\boldsymbol{a} \in \mathcal{A}$

$$\int_0^\infty \partial_a r(z, \boldsymbol{a}) \left\{(1 + \boldsymbol{c}) - e^{\gamma_k r(z,\boldsymbol{a})}\right\} F_k(dz) = 0$$

and denote by $\boldsymbol{\alpha_k^\dagger}[\boldsymbol{c}]$ its solution, if it exists.

Subject to a convexity condition, the process $\boldsymbol{\alpha_{t,k}^*} := \boldsymbol{\alpha_k^\dagger}[\boldsymbol{\eta_{t,k}}]$, $t \in [0, T]$, is the optimal insurer-k's control in feedback form.

Reinsurer's probability measure

Definition (Reinsurer's compensator)

An admissible compensator for the reinsurer is a nonnegative, $\mathbb{F}$-predictable random field $\varsigma = (\varsigma_t(\cdot))_{t\in[0,T]}$ such that, for all $t \in [0,T]$, $\varsigma_t(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ and

$$\mathbb{E}^{\mathbb{P}_k}\left[\exp\left(\int_0^T \int_{\mathbb{R}} \left[\frac{1-\varsigma_t(z)}{v^k(z)}\right]^2 N(dz,dt)\right)\right] < \infty, \qquad \forall k \in \mathcal{N}\,.$$

We denote the set of admissible compensators by $\mathcal{V}$.

Radon-Nikodym derivative from $\mathbb{P}_k$ to $\mathbb{Q}^{\varsigma}$:

$$\frac{d\mathbb{Q}^{\varsigma}}{d\mathbb{P}_k} := \exp\left(\int_0^T \int_{\mathbb{R}} \log\left(\frac{\varsigma_t(z)}{v^k(z)}\right) N(dz,dt) - \int_0^T \int_{\mathbb{R}} \left[\frac{\varsigma_t(z)}{v^k(z)} - 1\right] v^k(z)\, dz\, dt\right)$$

Reinsurer's Optimization Problem

Let insurer-k's demand for reinsurance be parameterised by $\alpha_k^* = \alpha^\dagger[\eta_{t,k}]$. The reinsurer seeks the contract parameters to attain

$$\sup_{\eta \in \mathfrak{C}} \inf_{\varsigma \in \mathcal{V}} \mathbb{E}^{\mathbb{Q}^\varsigma}\left[Y_T + \frac{1}{\varepsilon} \sum_{k \in \mathcal{N}} \pi_k D_{\mathrm{KL}}(\mathbb{Q}^\varsigma \,\|\, \mathbb{P}_k) \right],$$

where

- ε represents the reinsurer's overall ambiguity aversion,
- $\pi_k \geq 0$, $k \in \mathcal{N}$ are weights satisfying $\sum_{k \in \mathcal{N}} \pi_k = 1$.

Theorem

The Stackelberg equilibrium is $(\boldsymbol{\alpha}^*, \boldsymbol{\eta}^*, \varsigma^*)$ where $\boldsymbol{\alpha}^* = (\alpha_{\mathbf{1}}^{\dagger}[\eta_{\mathbf{1}}^*], \ldots, \alpha_{\boldsymbol{n}}^{\dagger}[\eta_{\boldsymbol{n}}^*])$ and $\boldsymbol{\eta}^* = (\eta_{\mathbf{1}}^*, \ldots, \eta_{\boldsymbol{n}}^*)$ are constants that satisfy a system of non-linear algebraic equations.

The optimal compensator is

$$\varsigma^*(z, \boldsymbol{\alpha}^*) = \exp\left\{\varepsilon \sum_{k \in \mathcal{N}} \big(z - r(z, \alpha_{\boldsymbol{k}}^*)\big)\right\} \prod_{\ell \in \mathcal{N}} v_\ell(z)^{\pi_\ell} .$$

Example: Excess-of-loss reinsurance

Two insurers with exponentially distributed losses

Suppose each insurer's loss is exponentially distributed with scale parameter $\xi_k \in (0, \min\{\frac{1}{\gamma_k}, \frac{1}{2\varepsilon}\})$, and that they each have intensity $\lambda_k > 0$, for $k = 1, 2$.

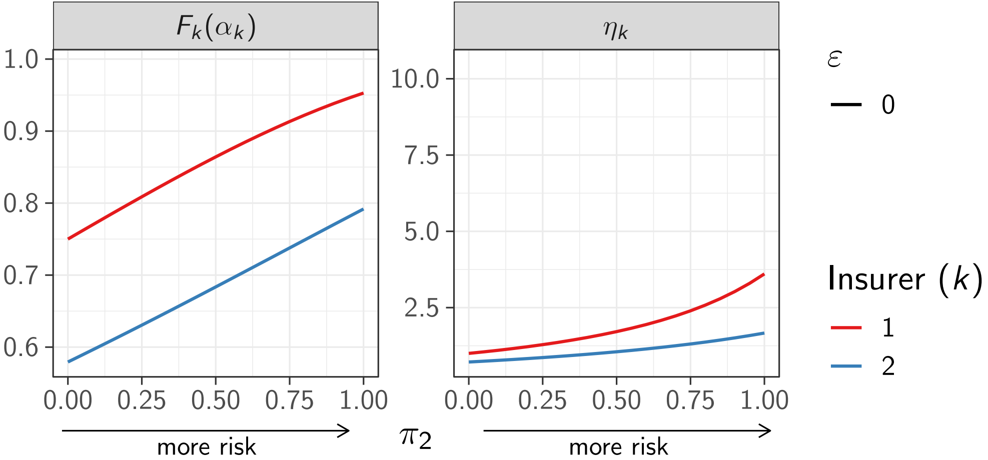
Then the Stackelberg equilibrium is given by

$$\varsigma^*(z) = v^g(z) \exp\left(\varepsilon \left[(z - \alpha_1^*)_+ + (z - \alpha_2^*)_+\right]\right),$$

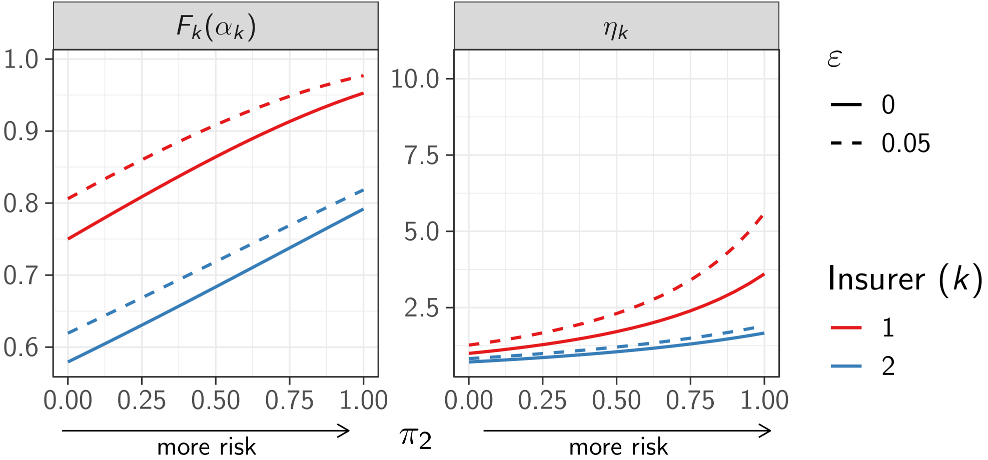
$$\alpha_k^* = \frac{1}{\gamma_k} \log\left(\frac{\int_{\alpha_k^*}^{\infty} \varsigma^*(z) dz}{\lambda_k\, e^{-\alpha_k^*/\xi_k} [1 - \gamma_k\, \xi_k]}\right), \qquad k = 1, 2,$$

$$\eta_k^* = e^{\gamma_k \alpha_k^*} - 1, \qquad k = 1, 2,$$

where v^g is the weighted geometric mean of the insurers' compensators.



Parameters: $\gamma_1 = \gamma_2 = 0.5, \quad \xi_1 = 1, \lambda_1 = 2, \quad \xi_2 = 1.25, \lambda_2 = 2.5$.



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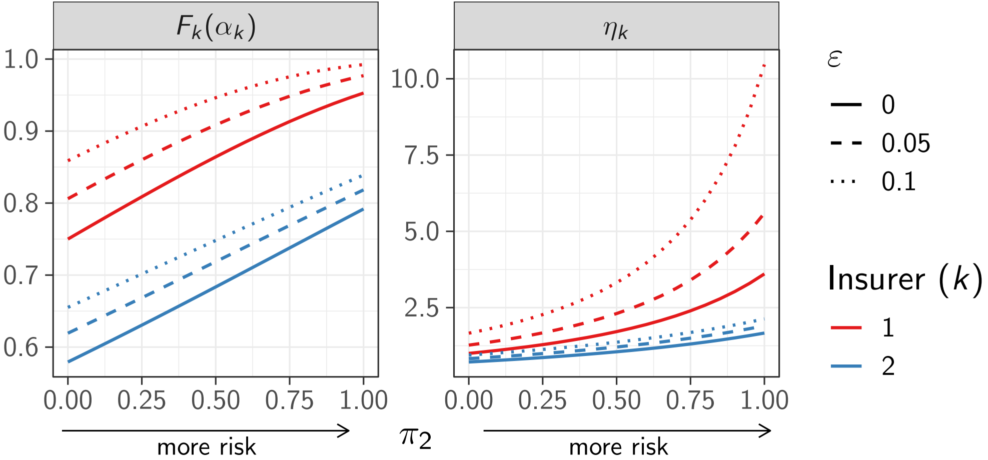

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Thank you for your attention!

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