Model Ambiguity in Risk Sharing with Monotone Mean-Variance

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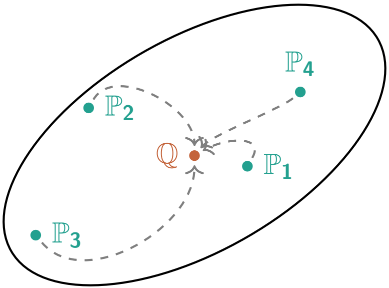
Introduction



An agent has multiple models/probability measures $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4$.

The agent has to make a decision optimally accounting for ambiguity about these models.

Agent must choose a model $\mathbb{Q}$ to optimize under.

In our setting:

- agent = insurer
- decision = risk sharing
- penalization = monotone mean variance, i.e., chi-squared penalty

Introduction

- **Insurer** in a non-life insurance market faces insurance losses over a finite horizon $[0, T]$.
- **Insurer** can share their risk with another agent, the **counterparty**, by ceding them a portion of their loss in return for a premium payment.
- Insurer has multiple models for the loss distribution: $\mathbb{P}_{\mathbf{1}}, \ldots, \mathbb{P}_{\boldsymbol{n}}, \mathbb{P}_{\boldsymbol{C}}$ and chooses a model $\mathbb{Q}$ to optimize the risk sharing under; counterparty sets premium under their model, $\mathbb{P}_{\boldsymbol{C}}$.

Probabilistic set-up

- Assume a complete, filtered measurable space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]})$ and $n+1$ equivalent probability measures $\mathbb{P}_{\mathbf{1}}, \dots, \mathbb{P}_{\boldsymbol{n}}$, $\mathbb{P}_{\boldsymbol{C}}$
- $N(d\xi, dt)$ is a Poisson random measure driving the insurance losses in the market.
- Under a measure $\mathbb{P}_{\boldsymbol{k}}$ for $k \in \mathcal{I}$, $\mathcal{I} := \{1, \dots, n, C\}$, N has $\mathbb{P}_{\boldsymbol{k}}$-compensator $\nu_k(d\xi, dt) = \nu_k(d\xi)dt$.
- Define the $\mathbb{P}_{\boldsymbol{k}}$-compensated PRM by

$$\tilde{N}^{\mathbb{P}_{\boldsymbol{k}}}(d\xi, dt) = N(d\xi, dt) - \nu_k(d\xi)dt\,.$$

- Each compensator admits a density $v_k(\xi)$, i.e., $\nu_k(d\xi) = v_k(\xi)d\xi$ for $k \in \mathcal{I}$.

Probabilistic set-up

Assumptions

$$\int_{\mathbb{R}_+} \frac{v_C^2(\xi)}{v_k(\xi)}\, d\xi < \infty \ \text{ for } k \in \mathcal{I}\,, \quad \int_{\mathbb{R}_+} \frac{v_C^3(\xi)}{v_j(\xi) v_k(\xi)}\, d\xi < \infty \ \text{ for } j,k \in \mathcal{I}\,.$$

Example

- Assume that $\nu_k(d\xi)$ is compound Poisson such that $v_k(\xi) = \lambda_k f_k(\xi)$, where $\lambda_k > 0$ and f_k is the density of a Gamma distribution with shape $m_k > 0$ and scale $\phi_k > 0$,
- The first assumption is satisfied if $2m_C > m_k$ and $2\phi_k > \phi_C$ for all $k \in \mathcal{I} \setminus \{C\}$.
- The second assumption is satisfied if for all $j,k \in \mathcal{I} \setminus \{C\}$, $3m_C > m_j + m_k$ and $3\phi_j\phi_k > \phi_C(\phi_j + \phi_k)$.

Insurer's surplus

- The insurer's wealth process follows a Cramér-Lundberg model with constant premium rate $c > 0$:

$$X_t^{CL} = x + ct - \int_0^t \int_{\mathbb{R}_+} \xi \, N(d\xi, ds) \, .$$

- Insurer cedes a portion $\alpha_t(\xi)$ of the loss $\xi \in \mathbb{R}_+$ to the counterparty.

Definition: admissible risk sharing strategies

We define the set of admissible risk sharing strategies, $\mathcal{A}$, as those strategies α_t that are $\mathbb{F}$-predictable random fields satisfying for $t \in [0, T]$,

$$\mathbb{E}^{\mathbb{P}_c} \left[\int_0^t \int_{\mathbb{R}_+} |\alpha_s(\xi)|^2 \, \nu_C(d\xi) \, ds \right] < \infty \quad \text{and}$$

$$\mathbb{E}^{\mathbb{P}_c} \left[\int_0^t \int_{\mathbb{R}_+} [\xi - \alpha_s(\xi)]^2 \, \nu_C(d\xi) \, ds \right] < \infty \, .$$

Insurer's surplus

- The counterparty charges the expected value premium principle with safety loading $\eta > 0$: $(1+\eta)\int_{\mathbb{R}_+} \alpha_t(\xi)\,\nu_C(d\xi)$.
- Assume that the risk sharing premium is such that $c < (1+\eta)\int_0^\infty \xi\,\nu_C(d\xi)$.
- The insurer's wealth process $X := (X_t)_{t\in[0,T]}$ is

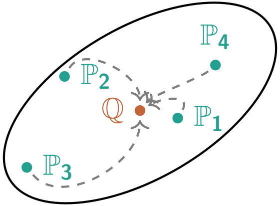
$$dX_t^{\alpha} = \Bigg[\underbrace{\quad c \quad}_{\text{insurer's premium}} - \underbrace{(1+\eta)\int_{\mathbb{R}_+} \alpha_t(\xi)\,\nu_C(d\xi)}_{\text{counterparty's premium}}\Bigg]dt - \underbrace{\int_{\mathbb{R}_+} [\xi - \alpha_t(\xi)]\, N(d\xi, dt)}_{\text{losses retained by insurer}}.$$

Monotone mean variance with model ambiguity

Recall: the insurer has $n+1$ models/probability measures $\mathbb{P}_{\mathbf{1}}, \dots, \mathbb{P}_{\boldsymbol{n}}, \mathbb{P}_{\boldsymbol{C}}$.

Insurer penalizes model ambiguity using the χ^2-divergence:

$$\chi^2(\mathbb{Q} \,\|\, \mathbb{P}) := \mathbb{E}^{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2 - 1\right].$$



Optimization Problem

The insurer seeks the solution to the following problem:

$$\sup_{\boldsymbol{\alpha} \in \mathcal{A}} \inf_{\mathbb{Q} \in \Delta^2} \left(\mathbb{E}^{\mathbb{Q}}[X_T^{\boldsymbol{\alpha}}] + \frac{1}{2\theta} \sum_{k \in \mathcal{I}} \pi_k \, \mathbb{E}^{\mathbb{P}_k}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}_{\boldsymbol{k}}}\right)^2 - 1\right]\right),$$

where $\theta > 0$ and $\pi_k \geq 0$, $k \in \mathcal{I} := \{1, \dots, n, C\}$ are given weights such that $\sum_{k \in \mathcal{I}} \pi_k = 1$.

The monotone mean-variance criterion [Maccheroni et al., 2009]

$$J_\theta^{MV}[X] = \mathbb{E}^{\mathbb{P}}[X] - \frac{\theta}{2}\mathrm{Var}^{\mathbb{P}}(X)$$

$$J_\theta^{MMV}[X] := \min_{\mathbb{Q}\in\Delta^2(\mathbb{P})} \left(\mathbb{E}^{\mathbb{Q}}[X] + \frac{1}{2\theta}\mathbb{E}^{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2 - 1 \right] \right)$$

where $\Delta^2(\mathbb{P}) = \{\mathbb{Q} \ll \mathbb{P} : \mathbb{E}^{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2\right] < \infty\}$.

Properties of MMV [Maccheroni et al., 2009]

- Agrees with MV criterion where it is monotone
- Best possible monotone approximation of the MV criterion outside of where it is monotone
- Unlike MV, MMV preserves second-order stochastic dominance

Related literature

- Recent approaches to mean-variance problems in insurance:
  - Sub-game Nash perfect equilibrium approach: [D. Li et al., 2017], [Chen and Shen, 2019], [Chen et al., 2021]
  - Time consistency using an auxiliary process: [Shen and Zou, 2021]
- Monotone mean-variance in optimal investment/insurance problems:
  - Stochastic factor model: [Trybuła and Zawisza, 2019], [Y. Li et al., 2024]
  - Constrained MMV: [Shen and Zou, 2022], [Hu et al., 2023]
  - MMV in insurance: [B. Li and Guo, 2021], [B. Li et al., 2024], [Shi and Xu, 2024]
- Multiple models
  - Optimal reinsurance: [Kroell et al., 2024]
  - Diffusion setting: [Jaimungal and Pesenti, 2024]

Chapter 4: Optimization Problem

Optimization Problem

The insurer seeks the solution to the following problem:

$$\sup_{\boldsymbol{\alpha}\in\mathcal{A}}\ \inf_{\mathbb{Q}\in\Delta^2}\left(\mathbb{E}^{\mathbb{Q}}[X_T^{\boldsymbol{\alpha}}]+\frac{1}{2\theta}\sum_{k\in\mathcal{I}}\pi_k\,\mathbb{E}^{\mathbb{P}_{\boldsymbol{k}}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}_{\boldsymbol{k}}}\right)^2-1\right]\right),$$

where $\theta>0$ and $\pi_k\geq 0$, $k\in\mathcal{I}:=\{1,\ldots,n,C\}$ are given weights such that $\sum_{k\in\mathcal{I}}\pi_k=1$.

Chapter 4: Optimization Problem

Optimization Problem

The insurer seeks the solution to the following problem:

$$\sup_{\boldsymbol{\alpha} \in \mathcal{A}} \inf_{\boldsymbol{\beta} \in \mathfrak{B}} \mathbb{E}^{\mathbb{Q}_{\boldsymbol{\beta}}} \left[X_T^{\boldsymbol{\alpha}} + \frac{1}{2\theta} \sum_{k \in \mathcal{I}} \pi_k \left(Z_{k,T}^{\boldsymbol{\beta}} - 1 \right) \right],$$

where $\theta > 0$ and $\pi_k \geq 0$, $k \in \mathcal{I} := \{1, \dots, n, C\}$ are given weights such that $\sum_{k \in \mathcal{I}} \pi_k = 1$.

Radon-Nikodym derivatives:

Define the stochastic processes $\{Z_{k,t}^{\boldsymbol{\beta}}\}_{t \in [0,T], k \in \mathcal{I}}$, for all $k \in \mathcal{I}$:

$$dZ_{k,t}^{\boldsymbol{\beta}} = Z_{k,t^-}^{\boldsymbol{\beta}} \int_{\mathbb{R}_+} [v_k(\xi) - \boldsymbol{\beta_t}(\xi)] \, d\xi dt - Z_{k,t^-}^{\boldsymbol{\beta}} \int_{\mathbb{R}_+} \left[1 - \frac{\boldsymbol{\beta_t}(\xi)}{v_k(\xi)} \right] N(d\xi, dt) \,,$$

$$Z_{k,0}^{\boldsymbol{\beta}} = 1 \,.$$

Auxiliary processes

- Define $\mathcal{B}$ to be set the of $\mathbb{F}$-predictable random fields $\boldsymbol{\beta_t}(\xi)$ satisfying for $t \in [0, T]$ and for all $k \in \mathcal{I}$

$$\mathbb{E}^{\mathbb{P}_{\boldsymbol{k}}}\left[\int_0^t \int_{\mathbb{R}_+} \left[1 - \frac{\boldsymbol{\beta_s}(\xi)}{v_k(\xi)}\right]^2 \nu_k(d\xi)ds\right] < \infty \,.$$

Definition: admissible compensators

Let $\mathfrak{B}$ denote the processes $\boldsymbol{\beta} \in \mathcal{B}$ such that for all $k \in \mathcal{I}$,

$$\mathbb{E}^{\mathbb{P}_{\boldsymbol{k}}}\left[Z^{\boldsymbol{\beta}}_{k,T}\right] = 1 \quad \text{and} \quad \mathbb{E}^{\mathbb{P}_{\boldsymbol{k}}}\left[\left(Z^{\boldsymbol{\beta}}_{k,T}\right)^2\right] < \infty \,.$$

Theorem (Optimal Controls)

The optimal controls in feedback form are

$$\alpha^*(t,\xi,\boldsymbol{z}) = \xi - \frac{1}{\theta}\sum_{k\in\mathcal{I}} \pi_k\, z_k\, \ell_k(T-t)\left[(1+\eta)\frac{v_C(\xi)}{v_k(\xi)} - 1\right]$$

$$\beta^*(\xi) = (1+\eta)\, v_C(\xi)\,,$$

where

$$\ell_k(t) = \exp\left(t\int_{\mathbb{R}_+}\left[1-(1+\eta)\frac{v_C(\xi)}{v_k(\xi)}\right]^2 \nu_k(d\xi)\right)\,,$$

and the insurer's value function is

$$\Phi(t,x,\boldsymbol{z}) = x + \sum_{k\in\mathcal{I}}\frac{\pi_k}{2\theta}\, z_k\, \ell_k(T-t) - \frac{1}{2\theta} - \left[(1+\eta)\int_{\mathbb{R}_+}\xi\,\nu_C(d\xi) - c\right](T-t)\,.$$

Processes Under Optimal Controls

Proposition

For $t \in [0, T]$,

$$Z^*_{k,t} = \exp\left(t \int_{\mathbb{R}_+} [v_k(\xi) - (1+\eta) v_C(\xi)]\, d\xi + \int_0^t \int_{\mathbb{R}_+} \ln\left((1+\eta) \frac{v_C(\xi)}{v_k(\xi)} \right) N(d\xi, ds) \right), \; k \in \mathcal{I},$$

$$X^*_t = x + \left[c - (1+\eta) \int_{\mathbb{R}_+} \xi\, \nu_C(d\xi) \right] t + \frac{1}{\theta} \sum_{k \in \mathcal{I}} \pi_k\, \ell_k(T) \; \left[1 - \ell_k(-t) Z^*_{k,t} \right] .$$

Sketch of proof: optimal controls

α^*, β^*, J derived using the Hamilton-Jacobi-Bellman-Isaacs equation.

Are α^* and β^* admissible?

Lemma

For $k \in \mathcal{I}$: $\mathbb{E}^{\mathbb{P}_k}[Z^*_{k,T}] = 1\,, \quad \mathbb{E}^{\mathbb{P}_k}[(Z^*_{k,T})^2] = \ell_k(T) < \infty\,, \quad \mathbb{E}^{\mathbb{P}_C}[(Z^*_{k,T})^2] < \infty\,.$

Then we can show that:

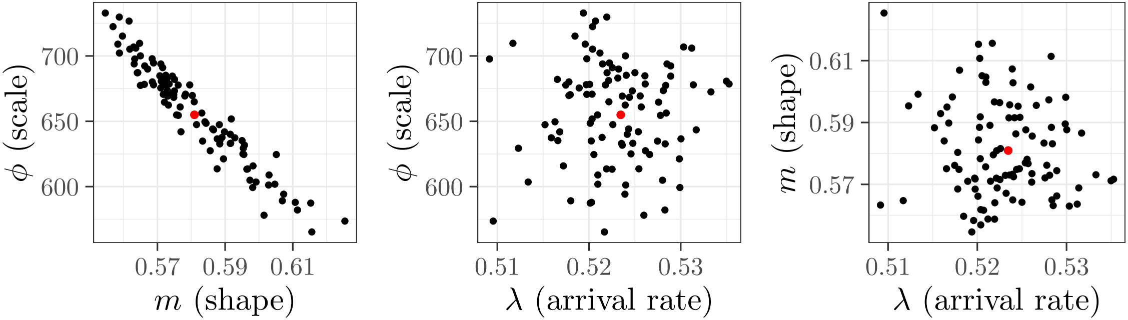
$$\left.\begin{aligned} \mathbb{E}^{\mathbb{P}_C}\left[\int_0^t \int_{\mathbb{R}_+} |\alpha^*(s,\xi,\boldsymbol{Z}^*_s)|^2\, \nu_C(d\xi)\, ds\right] < \infty \\ \mathbb{E}^{\mathbb{P}_C}\left[\int_0^t \int_{\mathbb{R}_+} [\xi - \alpha^*(s,\xi,\boldsymbol{Z}^*_s)]^2\, \nu_C(d\xi)\, ds\right] < \infty \end{aligned}\right\} \alpha^* \in \mathcal{A}$$

$$\left.\int_0^t \int_{\mathbb{R}_+} \left[1 - \frac{\beta^*_s(\xi)}{v_k(\xi)}\right]^2 \nu_k(d\xi) ds < \infty \right\} \beta^* \in \mathfrak{B}$$

Motivating data example

- Recent open-access insurance data set [Segura-Gisbert et al., 2024a,b], 105,555 observations, giving policy-level data on annual **motor insurance policies** of a Spanish non-life insurer for policies started in the years 2015–2018
- Using cross-validation, estimate 100 models $\mathbb{P}_{\boldsymbol{k}}$, $k = 1, \ldots, 100$ from the data set. For each estimate, we sample 50% of the data and then estimate the parameters.
- Assume that under all models $k \in \mathcal{I}$:
  - the claim arrival rate is Poisson distributed with rate $\lambda_k > 0$,
  - the severity distribution is Gamma distributed with shape parameter $m_k > 0$ and scale parameter $\theta_k > 0$.
- Estimate arrival rate and severity distribution by maximum likelihood.
- Estimate the counterparty's model, $\mathbb{P}_{\boldsymbol{C}}$, using the full dataset.

Estimated parameters

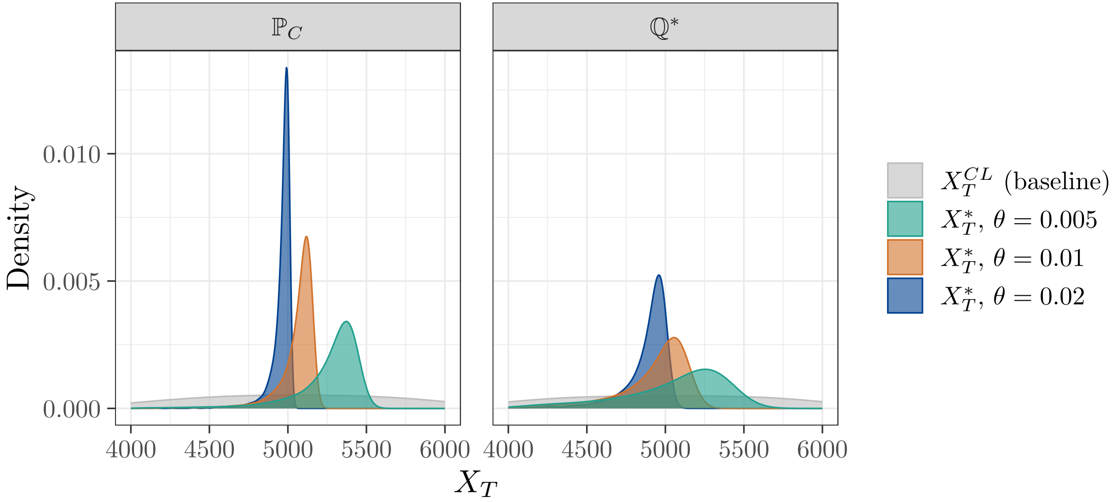


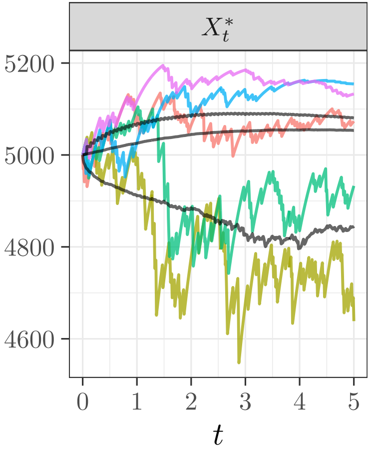
(a) Shape versus scale

(b) Arrival rate versus scale

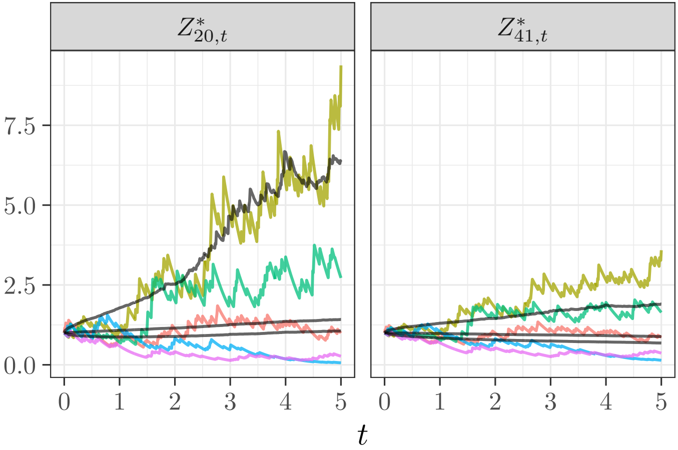
(c) Arrival rate versus shape

KDE of X_T under different scenarios





(a) Paths of X_t^* under $\mathbb{P}_c$



(b) Paths of $Z_{20,t}^*$, $Z_{41,t}^*$ under $\mathbb{P}_c$

One reference model

If there is only one model, $\mathbb{P}$, then the MMV criterion with model ambiguity reduces to the original MMV criterion [Maccheroni et al., 2009]

Proposition

The insurer's optimal controls are

$$\alpha^*(t, \xi, Z_t) = \xi - \frac{\eta}{\theta} e^{\lambda \eta^2 (T-t)} Z_t \,,$$
$$\beta^*(\xi) = (1 + \eta)\, v(\xi) \,.$$

Explicit solution

Proposition

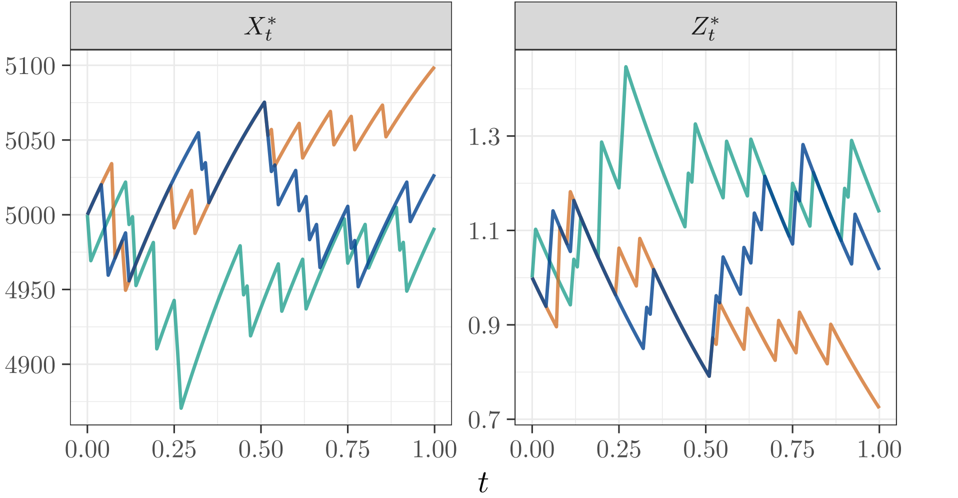
Let $M_t = \int_0^t \int_{\mathbb{R}} N(d\xi, dt)$. Then for $t \in [0, T]$,

$$Z_t = (1 + \eta)^{M_t} e^{-\eta\lambda t} ,$$
$$X_t = -\frac{1}{\theta} e^{\lambda\eta^2(T-t)} Z_t - (\eta - \phi)\lambda\mu t + \frac{1}{\theta} e^{\lambda\eta^2 T} .$$

Remark. For $t \in (0, T]$,

$$\mathrm{Corr}(X_t, Z_t) = -1 .$$

Paths: one reference model



Rewriting the optimal strategy

Recent work has shown that in many continuous-time investment problems, the **optimal strategies** for MMV and MV **coincide**.

- e.g., [Trybuła and Zawisza, 2019], [Strub and D. Li, 2020], [Shen and Zou, 2022], [Y. Li et al., 2024]

If the strategies coincide, we expect to be able to rewrite this to depend on $X_t - x$ or similar.

Restricting to one model: the optimal strategy is

$$\alpha^*(t,\xi,Z_t^*) = \xi - \frac{\eta}{\theta} e^{\lambda\eta^2(T-t)} Z_t^*, \alpha^*(t,\xi,X_t^*) = \xi - \eta \Bigg(-X_t^* + x + \Bigg[(1+\eta) \int_{\mathbb{R}_+} \xi \, \nu(d\xi) - c$$

Note: we cannot invert this relationship for X when there are multiple Zs.

Contributions

- We introduce a **new criterion** that generalizes the monotone mean-variance preferences to multiple reference models
- We derive **explicit solutions** for the insurer's **optimal risk-sharing strategy**, **optimal decision measure**, and their **wealth process**
- We prove that the strategy we obtain is admissible and that the value function satisfies the appropriate verification conditions
- We determine the **mean** and **variance** of the insurer's wealth process X, and show that the model penalization parameter θ penalizes the variance of X
- We illustrate the method with recent open-access non-life insurance data.

Thank you for your attention!

Download the pre-print:

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